Comments

Query. If $1^6 \equiv 1$ (mod7) and $2^6 \equiv 1$, $3^6 \equiv 1$, $4^6 \equiv 1$, $5^6 \equiv 1$, $6^6 \equiv 1$ (mod7) for all, is it true that $8^6 \equiv 1$(mod7), $9^6 \equiv 1$(mod7), etc.?
Yes, this holds for any number that is not itself a multiple of 7.
For example

$$9^6 = (7+2)^6 = 7^6 + 7.\ 7^5.2 + 21.\ 7^4.2^2 + \dots + 7.2^5+2^6 \equiv 2^6 \text{ (mod7)}$$

as all the terms before 2^6 are divisible by 7, that is, congruent to 0 (mod 7) and so we are left with 2^6 which we know is congruent to 1 (mod7).
The numbers 1-6 are called the ***least +ve residues*** (for 7).
Numbers 8,9,432 (non multiples of 7) are also residues but not *least* +ve ones.

Dedekind section/cut

The description of the Dedekind section of the real numbers given below, is that from the classic book "A course of Pure Mathematics" by the famous English mathematician G.H.Hardy.
Pages 1-27 give a description of the Dedekind section of the *rational numbers* (fractions in school language), which leads to the definition of a ***real number*** (e.g., $\sqrt{2}$).
Pages 28-29, in essence repeat the argument for real numbers, as for rational numbers, resulting, however, in no 'new' real numbers.

Numbers like $\sqrt{2}$ are ***algebraic numbers*** **but they are also real numbers** – we can think of them being measured on the real line, just like rationals such as 4/3 etc.
The algebraic numbers (which are also real numbers) are so named because they satisfy algebraic (polynomial) equations – polynomials with integer coefficients. For example $x^2 – 2 = 0$ is a quadratic (second-degree polynomial in x) which is satisfied by $x = \sqrt{2}$ and $x = -\sqrt{2}$, combined as $x = \pm\sqrt{2}$.
Then too, e.g., $x^3 – 2x^2 - 17x +34 = 0$ is a polynomial of degree 3 and has integer coefficients (namely -2, -17, -34).
It has the following roots, (solutions), $x = 2$, $x = \pm\sqrt{17}$.

Another example: $x^4 - 38x^2 + 225=0$ is a quartic polynomial with integer coefficients. One solution is $\sqrt{2} + \sqrt{17}$ an algebraic number.

Then too there are numbers which are NOT rational numbers or algebraic numbers **BUT ARE STILL REAL NUMBERS** – they can be measured on the real line. They are called ***transcendental numbers***.
Examples are π (pi), e, maybe γ (gamma) and multiples and exponents of these.

of integers or of real numbers, and that the values of the variables are the members of these classes.

Very Important.

17. Sections of the real numbers. In §§ 4–7 we considered 'sections' of the rational numbers, i.e. modes of division of the rational numbers (or of the positive rational numbers only) into two classes L and R possessing the following characteristic properties:

(i) that every number of the type considered belongs to one and only one of the two classes;

(ii) that both classes exist;

(iii) that any member of L is less than any member of R.

It is plainly possible to apply the same idea to the aggregate of all real numbers, and the process is, as the reader will find in later chapters, of very great importance.

Let us then suppose* that P and Q are two properties which are mutually exclusive, and one of which is possessed by every real number. Further let us suppose that any number which possesses P is less than any which possesses Q. We call the numbers which possess P the *lower* or *left-hand class* L, and those which possess Q the *upper* or *right-hand class* R.

Thus P might be $x \leqq \sqrt{2}$ and Q be $x > \sqrt{2}$. It is important to observe that a pair of properties which suffice to define a section of the rational numbers may not suffice to define one of the real numbers. This is so, for example, with the pair '$x < \sqrt{2}$' and '$x > \sqrt{2}$' or (if we confine ourselves to positive numbers) with '$x^2 < 2$' and '$x^2 > 2$'. Every rational number possesses one or other of the properties, but not every real number, since in either case $\sqrt{2}$ escapes classification.

There are now two possibilities†. Either L has a greatest member l, or R has a least member r. *Both* of these events cannot

* The discussion which follows is in many ways similar to that of § 6. We have not attempted to avoid a certain amount of repetition. The idea of a 'section', first brought into prominence in Dedekind's famous pamphlet *Stetigkeit und irrationale Zahlen*, is one which must be grasped by every reader of this book, even if he be one of those who prefer to omit the discussion of the notion of an irrational number contained in §§ 6–12.

† There were three in § 6.

occur. For if L had a greatest member l, and R a least member r, the number $\frac{1}{2}(l+r)$ would be greater than all members of L and less than all members of R, and so could not belong to either class. On the other hand *one* event must occur*.

For let L_1 and R_1 denote the classes formed from L and R by taking only the rational members of L and R. Then the classes L_1 and R_1 form a section of the rational numbers. There are now two cases to distinguish.

ie real no.

It may happen that L_1 has a greatest member α. In this case α must be also the greatest member of L. For if not we can find a greater, say β. There are rational numbers lying between α and β, and these, being less than β, belong to L, and therefore to L_1; and this is plainly a contradiction. Hence α is the greatest member of L. Very nice.

On the other hand it may happen that L_1 has no greatest member. In this case the section of the rational numbers formed by L_1 and R_1 is a real number α. This number α must belong to L or to R. If it belongs to L we can show, precisely as before, that it is the greatest member of L; and similarly that, if it belongs to R, it is the least member of R.

Thus in any case either L has a greatest member or R a least. Any section of the real numbers therefore 'corresponds' to a real number in the sense in which a section of the rational numbers sometimes, but not always, corresponds to a rational number. This conclusion is of very great importance; for it shows that the consideration of sections of all the real numbers does not lead to any further generalisation of our idea of number. Starting from the rational numbers, we found that the idea of a section of the rational numbers led us to a new conception of a number, that of a real number, more general than that of a rational number; and it might have been expected that the idea of a section of the real numbers would have led us to a conception more general still. The discussion which precedes shows that this is not the case, and that the aggregate of real numbers, or the continuum

* This was not the case in § 6.